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Modeling multi-country mortality dependence and its application in pricing survivor index swaps—A dynamic copula approach

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ABSTRACT

This paper introduces mortality dependence in multi-country mortality modeling using a dynamic copula approach. Specifically, we use time-varying copula models to capture the mortality dependence structure across countries, examining both symmetric and asymmetric dependence structures. In addition, to capture the phenomenon of a heavy tail for the multi-country mortality index, we consider not only the setting of Gaussian innovations but also non-Gaussian innovations under the Lee–Carter framework model. As tests of the goodness of fit of different dynamic copula models, the pattern of mortality dependence, and the distribution of the innovations, we used empirical mortality data from Finland, France, the Netherlands, and Sweden. To understand the effect of mortality dependence on longevity derivatives, we also built a valuation framework for pricing a survivor index swap, then investigated the fair swap rates of a survivor swap numerically. We demonstrate that failing to consider the dynamic copula mortality model and non-Gaussian innovations would lead to serious underestimations of the swap rates and loss reserves.

1. Introduction

As life expectancy increases dramatically, longevity risk has become an increasingly important consideration for defined benefit (DB) pension plans and annuity providers. Managing longevity risk using capital market solutions such as mortality-linked securities or derivatives has received great attention in recent years. The EIB/BNP longevity bond was the first proposed longevity bond, introduced in November 2004, though it was never issued. The q-forward contract between JPMorgan and the UK company Lucida was the world's first capital market derivative transaction, which took place in January 2008; the initial capital market longevity swap, executed in July 2008, enabled Canada Life to hedge its UK-based annuity policies. The first capital market Kortis longevity bond, executed in December 2010, was announced by the Swiss Re Reinsurance Company Ltd. In assessing these and other examples, much research has been devoted to the design and pricing of mortality-linked securities (Blake et al., 2006; Dowd et al., 2006; Biffis and Blake, 2009; Blake et al., 2010; Biffis et al., 2011; Huang et al., 2011; Wang et al., 2011; Wang and Yang, 2013).

As a potential means to increase hedge effectiveness, basis risk has been a primary concern in recent market developments. Basis risk results when the mortality experience of the index in longevity securities differs from the longevity risk exposure of the pension plan or annuity portfolio, when hedging longevity risk with the securities based on longevity indices. Dowd et al. (2006) point out that a hedge is only as good as the reference index based on the insurer's own mortality experience. Coughlan et al. (2011) also caution that basis risk can result in an imperfect longevity hedge, leaving some residual amount of risk. It is thus essential to evaluate the extent of this risk and weight the degree of risk reduction against the cost of the hedge. As Blake et al. (2012) explain, one of the reasons the EIB longevity bond never launched was because the basis risk in the bond appeared too great. It might have provided a reasonable hedge for male pension plan members in their 60s, but pension plans also include male members in their 70s and 80s, as well as women. The Swiss Re Mortality Bond, issued in 2003, reflects a combined mortality index instead, including France, England, the United States, Italy, and Switzerland. The

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mortality bond issued by Nathan Ltd. in 2008 similarly depends on a combined mortality index, for four countries: the United States, the United Kingdom, Canada, and Germany. Therefore, finding ways to deal with the basis risk through a longevity hedge is a critical question; it is unacceptable to ignore the basis risk when designing mortality-linked securities.

The mortality index underlying a mortality-linked security is particularly important for dealing with basis risk and increasing hedge effectiveness. We introduce a survivor index swap that can transfer the longevity risk pool across different insurers in different countries. Survivor swaps have been explored widely in prior literature (Dawson, 2002; Blake, 2003; Lin and Cox, 2005; Dowd et al., 2006), but existing studies do not deal with basis risk or the structure of a survivor swap that is based on a single population group. Dawson et al. (2010) first introduce the concept of a basis swap but ignore the associated pricing problem. In response, the current research attempts to model mortality dependence across countries, to deal with basis risk and build a valuation framework for pricing a survivor index swap.

This pricing decision depends on the dependence structure of the mortality rates across countries. Cox et al. (2006) investigate the Swiss Re mortality bond by assuming that its bond payments are contingent on the dynamics of two-country populations. Yang et al. (2011) offer a coherent mortality model for two populations within a Lee–Carter framework. Li and Hardy (2011) instead consider four extensions to the Lee–Carter model to measure basis risk in longevity hedges and demonstrate their findings with two populations.[1] In combination, these studies highlight the importance of mortality modeling for two populations, rather than a single one, when dealing with basis risk. However, to make better use of hedging effectiveness, the mortality model also should build on a multi-country framework that can capture mortality dependence across countries. To the best of our knowledge, Yang and Wang (2013) provided the first attempt to capture multi-country mortality dynamics; they applied a co-integration analysis with Gaussian residuals to investigate long-run equilibrium in a mortality time index. However, short-term catastrophe mortality shocks, such as the influenza pandemic in 1918, World Wars I and II, or the tsunami in December 2004, may lead to a co-jump in the mortality rates in different countries. It is thus crucial to address mortality jumps when modeling multi-country mortality dynamics. In addition, mortality dependence may be asymmetric or time-varying. For example, mortality dependence across countries may be lower during a smooth mortality evolution and higher during a huge mortality shock, such as the 2004 tsunami. To extend existing research that includes multi-country populations, we consider dynamic copula models and seek to capture mortality dependence that is allowed to be systematic or asymmetric in a multi-mortality framework, with both Gaussian and non-Gaussian residuals, to price a survivor index swap.

Copula models, which represent an appealing alternative to the Gaussian dependence structure, can help construct flexible, non-standard multivariate distributions and capture dependence between the variables. In finance literature, copula models have been used widely to capture dependence structures in financial data. For example, Engle (2002) addresses correlations among asset returns that are not constant over time, in a phenomenon known as asymmetric dependence, such that returns exhibit greater correlation during market downturns (high-volatility regime) than during market upturns (low-volatility regime) (Erb et al., 1994; Ramchand and Susmel, 1998; Longin and Solnik, 2001; Ang and Chen, 2002; Ang and Bekaert, 2002; Patton, 2004; Jondeau and Rockinger, 2006; Okimoto, 2008). The copula models also have been extended, to be time-varying. Manner and Reznikova (2010) study different time-varying copula models, including three popular methods, namely, copulas with time-varying parameters as proposed by Patton (2006), structural breaks in the copula parameters as proposed by Dias and Embrechts (2004), and the regime-switching (RS) copulas proposed by Pelletier (2006). All three dynamic copula methods have been used recently to deal with dependence structures, such as those underlying interest rates (Bu et al., 2011; Kumar and Okimoto, 2011), exchange rates (Dias and Embrechts, 2004; Patton, 2006), real estate securities (Zhou and Gao, 2012; Hoesli and Reka, 2013), and crude oil prices (Reboredo, 2011).

To deal with multi-country longevity risk, we need a mortality model to project the future mortality rates for different countries simultaneously. We extend the Lee–Carter model (Lee and Carter, 1992) to a multi-country framework, but to capture the phenomenon of a heavy tail for the multi-country mortality index, we also consider both Gaussian and non-Gaussian innovations, including the jump diffusion (JD) and generalized hyperbolic (GH) innovations for the Lee–Carter model, and we extend Wang et al.'s (2011) single-population approach.[2] The GH distribution (Barndorff-Nielsen, 1977) provides a flexible tool to model the empirical distribution with skewness, leptokurtosis and tail-thickness, and it nests many well-known distributions such as the normal, Student's t and hyperbolic (HYP) distributions (Barndorff-Nielsen and Blæsild, 1981); variance gamma (VG) distributions (Madan and Seneta, 1987, 1990); normal inverse Gaussian (NIG) distributions (Barndorff-Nielsen, 1995); and the GH-skewed t (GHST) distribution (Prause, 1999; Aas and Haff, 2006). We employ empirical mortality experience to fit the time-varying copula models and examine mortality dependence across countries. Because of the fundamental importance of the notion of linear correlation in finance and insurance, the time-varying copula models may have a non-trivial impact on the pricing of longevity securitization or the risk measurement of such positions. In our application, we provide the fair values of a survivor index swap and their value at risk (VaR) and conditional tail expectation (CTE). Finally, using the calibrated marginal and copula parameters of long-term multi-country mortality data, we provide a numerical analysis that demonstrates the asymmetric dependence phenomenon in multi-country mortality data and allows us to assess the impacts of the non-normal property and asymmetric multi-country mortality dependence on the pricing and risk management of a survivor index swap.

The contributions of this research are fivefold. First, we use time-varying copula models to introduce mortality dependence across countries in stochastic mortality modeling. Second, this article extends the original Lee–Carter model to a multi-country framework and considers both JD and GH innovations. Third, we carry out an empirical analysis and demonstrate the asymmetric dependence phenomenon in multi-country mortality data, as well as assess the impacts of the non-normal property and asymmetric multi-country mortality dependence. Fourth, as an application, we build a valuation framework for a survivor index swap using the Wang transform; in building this framework, we consider the basis

[1] The four extensions are as follows: both populations are jointly driven by the same single time-varying index, the two populations are co-integrated, the populations depend on a common age factor, or there is an augmented common factor model in which a population-specific time-varying index is added to the common factor model, with the property that it will tend toward a certain constant level over time.

[2] Wang et al. (2011) adopt five non-Gaussian distributions to model the error terms of the Lee–Carter model: Student's t-distribution, generalized hyperbolic skewed Student's t-distribution, jump diffusion distribution, variance gamma distribution, and normal inverse Gaussian distribution.

risk and currency risk across countries. Fifth, we numerically assess the value at risk (VaR) and conditional tail expectation (CTE) of the basis swaps to evaluate hedging effectiveness.

The remainder of this paper is organized as follows. In Section 2, we present a Lee–Carter model with a multi-country framework, then implement a dynamic copula approach to capture the dependence of mortality rates across countries. In addition to the dynamic copula function and the estimation, we introduce the Gaussian and non-Gaussian models for mortality indices. The structure and the valuation framework for a survivor index swap are the focus in Section 3. In Section 4, we fit the classical LC model with the dynamic copula and non-Gaussian residuals, according to the empirical data. We also examine the impact of mortality dependence on a survivor swap by calculating the fair swap premium, along with the VaR and CTE of the survivor index swaps. On the basis of a numerical analysis of the valuation of a survivor index swap, we draw some key conclusions and detail the implications.

2. Multi-country mortality modeling under a dynamic copula framework

To price the survival index swaps, we first deal with multi-country mortality dynamics and consider mortality dependence across countries. We build the multi-country mortality dynamics by employing the Lee–Carter framework and use the dynamic copula models to capture the mortality dependence. In addition, we employ the JD and GH distributions to model the residuals of the mortality indices.

2.1. The Lee–Carter model under a multi-country framework

We employ the Lee–Carter model and extend their model to a multi-country framework. We analyze changes in mortality as a function of both age x and time t on a filtered probability space $\left(\Omega,\ \Im,\ P,\ (\Im_s)_{s=0}^{t+T}\right)$, where P is the physical probability measure and $\Im_t$ is the information available at time t. If we consider N population groups from different countries, then $m_{x,\ t}^j,\ j = 1, \ldots, N$, is the mortality force at age x in the jth population group during calendar year t. That is,

$$\ln m_{x,\ t}^j = a_x^j + b_x^j k_t^j + e_{x,\ t}^j, \quad j = 1, \ldots, N, \tag{1}$$

where the Lee–Carter model can capture the age–period effect for the jth population with a_x^j coefficients and the force of mortality for the jth population changes according to an overall mortality index k_t^j, which is modulated by an age response b_x^j. The error term $e_{x,\ t}^j$ reflects a particular, age-specific historical influence that is not captured by the model. In addition, the parameters b_x^j and k_t^j are subject to $\sum_x^{b_x^j} = 1$ and $\sum_t^{k_t^j} = 0$ where these two conditions ensure the model identification. We use the approximation method to fit the parameters a_x^j, b_x^j and k_t^j for each set of mortality data. According to the two constraint conditions $\sum_x^{b_x^j} = 1$ and $\sum_t^{k_t^j} = 0$, $\hat{a}_x^j$ is simply the average value over time of $\ln\left(m_{x,\ t}^j\right)$, and $\hat{k}_t^j$ is the sum over various ages of $\ln\left(m_{x,\ t}^j\right) - \hat{a}_x^j$. Regressing $\ln\left(m_{x,\ t}^j\right) - \hat{a}_x^j$ on $\hat{k}_t^j$, we can obtain $\hat{b}_x^j$ by using a simple regression model without an intercept parameter. Finally, following the two-step estimation procedure for k_t^j by Lee and Carter (1992), we re-estimate $\hat{k}_t^j$ in a second step, using actual number of deaths in different population groups, such that the estimated number of deaths is close to the actual number of deaths in the jth population group.

2.2. The Gaussian and non-Gaussian models for the mortality indices

To capture the phenomenon of a heavy tail for the multi-country mortality index, we consider not only the setting of Gaussian innovations but also non-Gaussian innovations including the jump diffusion (JD) and generalized hyperbolic (GH) innovations with the Lee–Carter model. The original Lee–Carter model assumes a Gaussian innovation for the mortality index and forecast the future dynamics of the mortality index k_t^j. In most applications, k_t^j is well-modeled as a random walk with drift or referred to as the ARIMA(0, 1, 0) model. However, Lee (2000) suggests that an added moving average term or autoregressive term may be superior. Consequently, we use a standard ARIMA(P, 1, Q) time-series model to describe the mortality forecast, as follows:

$$k_t^j - k_{t-1}^j = \omega + \sum_{h=1}^{P} \varphi_h \left(k_{t-h}^j - k_{t-h-1}^j\right) + \sum_{h=0}^{Q} \psi_h \varepsilon_{t-h}^j, \quad j = 1, \ldots, N, \tag{2}$$

where P and Q denote the AR and MA orders, respectively; ω, φ_h, and ψ_h are the drift, AR, and MA parameters, respectively, with $\psi_0 = 1$; and ε_t^j is a sequence of independent and identically random variables with zero mean.

Under non-Gaussian innovations for the mortality indices, error terms ε_t^j can be expressed as[3]:

$$\varepsilon_t^j = a_j + \sigma_j z_j + \sum_{i=1}^{N^j} Y_i^j, \tag{3}$$

where N^j is the Poisson distribution with intensity λ_j; each Y_i^j, independent of z^j and N^j, is a normal distribution with mean μ_j and standard deviation δ_j. When ε_t^j follows a JD distribution, the probability density function of the ε_t^j JD distribution is of the form:

$$f_{JD}\left(x|\, a_j, \sigma_j, \lambda_j, \mu_j, \delta_j\right) = \sum_{n=1}^{\infty} \frac{\lambda_j^n e^{-\lambda_j}}{n!} \Phi\left(x|\, a_j + n\mu_j, \sigma_j^2 + n\delta_j^2\right), \tag{4}$$

where $\Phi\left(\cdot|\, \tilde{\mu}, \tilde{\sigma}^2\right)$ is a normal probability density function with mean $\tilde{\mu}$ and variance $\tilde{\sigma}^2$.

When ε_t^j follows a GH distribution, the probability density function of the ε_t^j GH distribution is of the form:

$$f_{GH}\left(x|\, \alpha_j, \beta_j, \delta_j, \gamma_j, \theta_j\right) = \frac{\left(\frac{\sqrt{\alpha_j^2 - \beta_j^2}}{\delta_j}\right)^{\gamma}}{\sqrt{2\pi}\left(K_{\gamma_j}\left(\delta_j\sqrt{\alpha_j^2 - \beta_j^2}\right)\right)} \times e^{\beta_j(x-\theta_j)} \frac{K_{\gamma_j - \frac{1}{2}}\left(\alpha_j\sqrt{\delta_j^2 + \left(x - \theta_j\right)^2}\right)}{\left(\frac{\sqrt{\delta_j^2 + (x-\theta_j)^2}}{\alpha_j}\right)^{\frac{1}{2} - \gamma_j}}, \tag{5}$$

where K_γ is the modified Bessel function of the second kind with index γ_j; δ_j is the scale parameter; θ_j is the shift parameter; and γ_j, α_j and β_j determine the shape of the GH distribution. These parameters obey the following constraints: $\alpha_j > \left|\beta_j\right| \geq 0$ and $\gamma_j \in R$. To ensure the zero-mean condition, we have

$$\theta_j = -\beta_j \delta_j K_{\gamma_j+1}\left(\delta_j\sqrt{\alpha_j^2 - \beta_j^2}\right) / \left(\sqrt{\alpha_j^2 - \beta_j^2} K_{\gamma_j}\left(\delta_j\sqrt{\alpha_j^2 - \beta_j^2}\right)\right). \tag{6}$$

[3] Let $a_j = -\lambda_j \mu_j$ in the JD model to ensure that the mean of the error terms equals 0.

Table 1
The special cases of GH distributions.

	α_j	β_j	γ_j	Notes
Generalized Hyperbolic (GH)	>0	$\sim$	$\sim$	
Hyperbolic (HYP)	>0	$\sim$	1	
Variance Gamma (VG)	>0	$\sim$	>0	for $\delta_j = 0$
Normal Inverse Gaussian (NIG)	>0	$\sim$	-0.5	
Skewed t (GHST)	>0 and $\alpha_j \to \lvert\beta_j\rvert$	$\sim$	<0 and $\gamma_j = -\frac{v_j}{2}$	v_j is the degree of freedom of GHST
Student's t (T)	0	0	<0 and $\gamma_j = -\frac{v_j}{2}$	for $\delta_j = \sqrt{v_j}$, v_j is the degree of freedom of T

Note: The symbol $\sim$ denotes the free parameters in the corresponding distributions.

The GH distribution nests several well-known sub-distributions including HYP distribution, VG distribution, NIG distribution, GHST distribution and Student's t (T) distribution. In this paper, we also model the law of ε_t^j as the family of GH distributions, and the corresponding parameters of each distribution are presented in Table 1.

2.3. The time-varying copula models

We use the time-varying copula model to capture the mortality dependence across countries. Copulas, introduced by Sklar (1959), are tools for modeling dependence between random variables. It links univariate distributions to the multivariate distribution of the related variables. Many studies demonstrate that the specification of time-varying correlation gives better results than unconditional copula models (e.g., Alexandra and Paul, 2010). Consequently, we also employ the time-varying copula to empirically test the mortality dependence structure.

To model the mortality dependence across N countries, under the time-varying copula model, the conditional N-dimensional cumulative distribution function (cdf) of ε_t^j is as follows:

$$\begin{aligned} &P\left(\varepsilon_t^1 \le x_1, \ldots, \varepsilon_t^N \le x_N \middle| \Im_{t-1}\right) = F\left(x_1, \ldots, x_N \middle| \Im_{t-1}\right) \\ &\quad = C\left(F_1\left(x_1\middle| \Im_{t-1}\right), \ldots, F_N\left(x_N\middle| \Im_{t-1}\right)\middle| \Im_{t-1}\right), \end{aligned} \tag{7}$$

where F_j is the marginal conditional cdf of ε_t^j and C is a conditional copula function. In view of Eq. (7), if the copula is sufficiently differentiable, the joint density function of ε_t^j, $j = 1, \ldots, N$, can be obtained as follows:

$$\begin{aligned} &f\left(x_1, \ldots, x_N\middle| \Im_{t-1}\right) \\ &\quad = c\left(F_1\left(x_1\middle| \Im_{t-1}\right), \ldots, F_N\left(x_N\middle| \Im_{t-1}\right)\middle| \Im_{t-1}\right) \prod_{j=1}^{N} f_j\left(x_j\middle| \Im_{t-1}\right), \end{aligned} \tag{8}$$

where f is the joint conditional probability density function (pdf); f_j is the marginal conditional pdf of ε_t^j; and c is a conditional copula density function that satisfies

$$c\left(u_1, u_2, \ldots, u_N\middle| \Im_{t-1}\right) = \frac{\partial^N C\left(u_1, u_2, \ldots, u_N\middle| \Im_{t-1}\right)}{\partial u_1 \ldots \partial u_N}. \tag{9}$$

We examine both the symmetric and asymmetric mortality dependence structure. Thus, we use the symmetric multivariate copulas – Gaussian copula and Student's t copula – to characterize the symmetric mortality dependence and the asymmetric copulas – the Gumbel copula, Clayton copula and skewed t copula – to characterize the asymmetric mortality dependence. We will empirically test whether the multi-country mortality indices display symmetry or asymmetry in the dependence structure.

The multivariate Gaussian copula is of the form:

$$C_G(u_1, \ldots, u_N; R) = \Phi_R(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_N)), \tag{10}$$

where Φ^{-1} denotes the inverse cumulative density of the standard normal; $\Phi_R(x_1, \ldots, x_N)$ denotes the standard multivariate normal cumulative distribution and R is a N-by-N correlation matrix. The Gaussian copula density is given by

$$c_G(u_1, \ldots, u_N; R) = |R|^{-1/2} \exp\left[-\frac{1}{2}(x'R^{-1}x - x'x)\right], \tag{11}$$

where $x = (\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_N))$ and $|R|$ is the determinant of the covariance R. The Gaussian copula has zero upper (right) and lower (left) tail dependence, that is $\lambda_U = \lambda_L = 0$.[4]

The multivariate Student's t copula is given by

$$C_T(u_1, \ldots, u_N; R, v) = T_{R,v}(T_v^{-1}(u_1), \ldots, T_v^{-1}(u_N)), \tag{12}$$

where R is a N-by-N correlation matrix; T_v^{-1} is the inverse of the cdf of the univariate standard Student's t with v degrees of freedom; and $T_{R,v}$ has the density of the form:

$$f_{R,v}(y_1, \ldots, y_N) = \frac{\Gamma\left(\frac{v+N}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\sqrt{(\pi v)^n |R|}} \left(1 + \frac{y'R^{-1}y}{v}\right)^{\frac{-v+N}{2}}, \tag{13}$$

where $y = (y_1, \ldots, y_N)'$. The corresponding Student's t copula density is given by

$$c_T(u_1, \ldots, u_N; R, v) = \frac{f_{R,v}(x_1, \ldots, x_N)}{\prod_{i=1}^{N} f_v(x_i)}, \tag{14}$$

where $x_i = T_v^{-1}(u_i)$ and $f_v(.)$ is the density of the Student's t distribution with v degrees of freedom with $v > 2$ for finite covariance.

The skew t copula, proposed by Demarta and McNeil (2005), is defined in the following representation:

$$\begin{aligned} &C_{ST}\left(u_1, \ldots, u_N; \sum, v, \eta\right) \\ &\quad = ST_{\sum,v,\eta}(ST_{v,\eta_1}^{-1}(u_1), \ldots, ST_{v,\eta_N}^{-1}(u_N)), \end{aligned} \tag{15}$$

where $v > 4$ for finite covariance; $\eta = (\eta_1, \ldots, \eta_N)'$ is a N-dimensional vector accounting for the skewness; $ST_{\sum,v,\eta}$ represents a multivariate skewed t distribution with the following density:

$$\begin{aligned} &f_{ST}\left(x_1, \ldots, x_N; \sum, v, \eta\right) \\ &\quad = c\frac{\exp\left(x'\overset{-1}{\sum}\eta\right) K_{\frac{v+N}{2}}\left(\sqrt{\left(v + x'\overset{-1}{\sum}x\right)\eta'\overset{-1}{\sum}\eta}\right)}{\left(\sqrt{\left(v + x'\overset{-1}{\sum}x\right)\eta'\overset{-1}{\sum}\eta}\right)^{-\frac{v+N}{2}}\left(1 + \frac{x'\overset{-1}{\sum}x}{v}\right)^{\frac{v+N}{2}}}, \end{aligned} \tag{16}$$

[4] The tail dependence coefficients are defined as $\lambda_L = \lim_{u\to 0} \frac{C(u,u)}{u}$ and $\lambda_U = \lim_{u\to 0} \frac{2u-1+C(1-u,1-u)}{u}$.

where

$$c=\frac{2^{1-\frac{v+N}{2}}}{\Gamma(v/2)\,(\pi v)^{\frac{N}{2}}\sqrt{\left|\sum\right|}},\tag{17}$$

and $ST^{-1}_{v,\eta_j}, j=1,\ldots,N$, is the inverse cdf of the jth margin with its density as follows:

$$f_{ST}(x_j;v,\eta_j)$$
$$=\frac{2^{1-\frac{v+1}{2}}}{\Gamma(v/2)\sqrt{\pi v}}\frac{\exp\left(x_j\eta_j\right)K_{\frac{v+1}{2}}\left(\sqrt{\left(v+x_j^2\right)\eta_j^2}\right)}{\left(\sqrt{\left(v+x_j^2\right)\eta_j^2}\right)^{-\frac{v+1}{2}}\left(1+\frac{x_j^2}{v}\right)^{\frac{v+1}{2}}}.\tag{18}$$

Note that when each $\eta_j\to 0$ for $j=1,\ldots,N$, the skewed t density in Eq. (16) converges to the multivariate Student's t density.

The Archimedean copulas can efficiently capture the tail dependences coming from the extreme observations that are caused by the asymmetry. The Gumbel copula, one of the Archimedean copulas, is defined as

$$C_G(u_1,\ldots,u_N;\theta)=\exp\left\{-\left[\left(\sum_{i=1}^{N}(-\ln u_i)^{\theta}\right)^{1/\theta}\right]\right\},\tag{19}$$

where $\theta\in[1,\infty)$ is the degree of dependence between $u_i, i=1,\ldots,N$. There is no dependence if θ approaches one and there is a fully dependent relationship if θ approaches infinity. The Gumbel copula has upper tail dependence and only allows for positive dependence.

The Clayton copula, one of the Archimedean copulas, is given by

$$C_C(u_1,u_2,\ldots,u_N;\theta)=\left(\sum_{i=1}^{N}u_i^{-\theta}-N+1\right)^{-1/\theta},\tag{20}$$

with $\theta\in(0,\infty)$. There is no dependence if θ approaches zero and there is a fully dependent relationship if θ approaches infinity. The Clayton copula only has lower tail dependence.

Engle (2002) propose a multivariate GARCH model with dynamic conditional correlation (DCC) that allows correlations to be driven by the cross product of the lagged standardized residuals and an autoregressive term. As shown by Vogiatzoglou (2010) and Manner and Reznikova (2010), this DCC framework can easily be adapted to model the dynamics copulas such as multivariate Gaussian copula, multivariate Student's t copula and multivariate skewed t copula. Let $Y_t=(Y_{1t},\ldots,Y_{Nt})'$ where $Y_{it}=\Phi^{-1}(u_{it})$ for a multivariate Gaussian copula, $Y_{it}=T_v^{-1}(u_{it})$ for a multivariate Student's t copula and $Y_{it}=ST^{-1}_{v,\eta_i}(u_{it})$ for a multivariate skewed t copula. The dynamic correlation matrix Σ_t in a DCC copula model is specified as

$$\Sigma_t=diag\left\{Q_t\right\}^{-\frac{1}{2}}Q_t diag\left\{Q_t\right\}^{-\frac{1}{2}},\tag{21}$$
$$Q_t=(1-\alpha-\beta)\overline{Q}+\alpha Y_{t-1}Y'_{t-1}+\beta Q_{t-1},\tag{22}$$

where $\overline{Q}$ is parameter correlation matrix and α and β are non-negative parameters satisfying $\alpha+\beta<1$. Therefore Σ_t is a dynamic correlation matrix as long as Q_t is positive definite. For more details of the DCC copula, the reader can refer to Serban et al. (2007), Vogiatzoglou (2010), and Manner and Reznikova (2010).

Patton (2006) proposes observation driven copula models for which time-varying dependence parameter is a function of transformations of the lagged data and an autoregressive term. For the non-Gaussian case such as one-parameter bivariate Archimedean copula family, as shown by Patton (2006) and Manner and Reznikova (2010), the evolution of a dependence parameter θ of a Archimedean copula is

$$\theta_t=\widetilde{\Lambda}_\theta\left\{\omega_\theta+\beta_\theta\widetilde{\Lambda}_\theta^{-1}(\theta_{t-1})+\gamma_\theta\frac{1}{10}\sum_{j=1}^{10}\left|u_{1,t-j}-u_{2,t-j}\right|\right\},\tag{23}$$

where $\widetilde{\Lambda}_\theta$ is the modified logistic transformation to ensure that the parameter always remains in its range. Note that Eq. (23) contains an autoregressive term $\widetilde{\Lambda}_\theta^{-1}(\theta_{t-1})$ to capture the persistence in the dependence parameter and the mean of the last 10 observations of the transformed lagged variables to capture any variation in dependence.

For one-parameter multivariate Clayton and Gumbel copulas, following the similar concept of Patton (2006), in this paper we extend Eq. (23) from the bivariate case to a multivariate case as follows:

$$\theta_t=\widetilde{\Lambda}_\theta\left\{\omega_\theta+\beta_\theta\widetilde{\Lambda}_\theta^{-1}(\theta_{t-1})\right.$$
$$\left.+\sum_{i>j}\gamma_{i,j}^{\theta}\left(\frac{1}{10}\sum_{h=1}^{10}\left|u_{i,t-h}-u_{j,t-h}\right|\right)\right\}.\tag{24}$$

Because $\theta_t>0$ for Clayton copula and $\theta_t>1$ for Gumbel copula, Manner and Reznikova (2010) suggest $\widetilde{\Lambda}_\theta(x)=\exp(x)$ for the Clayton copula and $\widetilde{\Lambda}_\theta(x)=\exp(x)+1$ for the Gumbel copula to ensure the parameter always remains in its domain.

2.4. Estimation of time-varying copulas

Denote the sample of observed data by $x_t=\{x_{1t},\ldots,x_{Nt}\}, t=1,\ldots,T$, is the realized residual vector in the tth year. The log likelihood function is given by:

$$L(\Theta)=\sum_{t=1}^{T}\log f(x_t|\Im_{t-1};\Theta),\tag{25}$$

where Θ is a vector including the parameters of the time-varying copula model, which includes the parameters of marginal pdfs and copula functions. Substituting Eq. (8) into Eq. (25) yields

$$L(\Theta)=\sum_{i=1}^{N}\sum_{t=1}^{T}\log f_i\left(x_{it}|\Im_{t-1};\Theta\right)$$
$$+\sum_{t=1}^{T}\log\left(c\left(F_1\left(x_{1t}|\Im_{t-1}\right),\ldots,\right.\right.$$
$$\left.\left.F_N\left(x_{Nt}|\Im_{t-1}\right)|\Im_{t-1};\Theta\right)\right).\tag{26}$$

The parameters can be estimated by maximum likelihood (ML) and the inference for the margins (IFM). Compared to ML, the IFM, a two-step estimation, is easily implemented and yields asymptotically efficient estimates (Joe, 1997; Patton, 2006). As a result, we use the IFM method that allows for a two-step estimation procedure. Therefore, we can estimate each set of marginal parameters separately in the first step. Or equivalently, the first step is then equivalent to N single estimations of univariate distributions. In the second step, given the calibrated marginal parameters in the first step, we can obtain the time-varying copula parameters. The detailed two-step estimation procedure is provided in the Appendix.

3. Valuation framework for survivor index swaps

To increase the hedge effectiveness, a security that can transfer longevity risk across countries is needed. We demonstrate with a survivor swap which the underlying survivor index is based on a weighted multi-country survivor probability. We refer such security as a survivor index swap in this research. The structure and the valuation framework for a survivor index swap are introduced below.

3.1. The structure of a survivor index swap

A survivor swap has been widely explored in prior literature (Dawson, 2002; Blake, 2003; Dowd, 2003; Lin and Cox, 2005). A survivor swap is an agreement that involves the periodic exchange of a series of preset payments for a series of random mortality-dependent payments. On each payment date, the fixed-rate payer pays a preset amount, equal to the value of the notional principal multiplied by a fixed rate, and receives in return from the floating-rate payer a random mortality-dependent payment, equal to the value of the notional principal multiplied by the unexpected shock in survival probability (i.e., the difference between the actual survival probability and the reference survival probability). Thus, survivor swaps can be used to transfer the unexpected shock in mortality improvement. Dowd et al. (2006) point out that hedge is only as good as it is when the reference index is based the insurer's own mortality experience. In addition, a single mortality benchmark has been considered to be inadequate to create an effective hedge. Blake et al. (2012) point out that one of the reasons why the EIB longevity[5] bond was not launched was because the basis risk in the bond was considered to be too great. To increase the hedging effectiveness, we extend the structure of a vanilla survivor swap analyzed by Dowd et al. (2006) and Dawson et al. (2010) to a survivor index swap that the underlying mortality index is linked to a benchmark cohort in different countries instead of a single population mortality index.

Assume that T-year survivor index swap is issued at time t_0. The underlying n-year survival index is a weighted survival probability in different populations based on the annuities issued to the cohort aged x_0 at time, which is denoted as $S_{x_0}(t_0, n)$. Especially, we consider the exchange rate converting the annuity into domestic currency in calculating the weight. Let $w_j(t_0)$ be the weight for the jth population group at time t_0, which can be expressed as

$$w_j(t_0) = \frac{L^j_{x_0} Q_j(t_0)}{\sum_{i=1}^{N} L^i_{x_0} Q_i(t_0)}, \quad j = 1, \ldots, N \tag{27}$$

where $L^j_{x_0}$ is the annuities issued to the cohort aged x_0 at time t_0 in the jth population group and $Q_j(t_0)$ is the exchange rate which is used to convert the jth foreign payout into domestic currency.

Thus, the underlying combined survivor index for the survivor index swap across N populations, $S_{x_0}(t_0, n)$, can be expressed as

$$S_{x_0}(t_0, n) = \sum_{j=1}^{N} w_j(t_0 + n) \times {}_n p^j_{x_0}, \quad n = 1, \ldots, T, \tag{28}$$

where ${}_n p^j_{x_0}$ denote the n-year survival probability of the jth population group aged x_0 that reaches age $x_0 + n$ in calendar year $t_0 + n$, which is calculated according the multi-country mortality dynamics. Then

$${}_n p^j_{x_0} = \exp\left(-\int_0^n m^j_{x_0+u,\, t_0+u}\, du\right),$$
$$j = 1, \ldots, N, \; n = 1, \ldots, T. \tag{29}$$

We assume that the mortality rates are constant within certain age and time windows but may vary from one window to the next. Specifically, given any integer age x and calendar year t, we assume that

$$m^j_{x+\xi,\, t+\tau} = m^j_{x,\, t} \quad \text{for } 0 \le \xi, \tau < 1, \; j = 1, \ldots, N,$$
$$n = 1, \ldots, T. \tag{30}$$

Thus, the n-year survival probability of the jth mortality group can be calculated as

$${}_n p^j_{x_0} = \exp\left(-\sum_{h=0}^{n-1} m^j_{x_0+h,\, t_0+h}\right),$$
$$j = 1, \ldots, N, \; n = 1, \ldots, T. \tag{31}$$

Let c_S denote the fixed swap rate and $c_{S\,n}$ be the floating swap rate for a unit notional principle. On each of the payment dates t, $t = 1, \ldots, T$, a fixed-rate payer of the survivor index swap pays the notional principal multiplied by a predetermined fixed proportion, $c_S = (1 + \pi) S^{ref}_{x_0}(t_0, n)$ to the floating-rate payer and in return receives the notional principal multiplied by a survivor index, $c_{S\,n} = S_{x_0}(t_0, n)$. $S^{ref}_{x_0}(t_0, n)$ is the reference survival index anticipated by using the best estimate of the Lee–Carter model to calculate the reference survival probability in the jth population group, ${}_n p^{ref,j}_{x_0}$. π is the swap premium which is determined by setting the initial value of the swap equal to zero for each party.

3.2. Risk-neutral survival probabilities

To change the real-world probability measure P to a risk-neutral probability, Wang (2000) proposes a distortion operator, called the Wang risk measure, with the following transformation:

$$\tilde{F}^j_t(x) = \Phi\left(\Phi^{-1}\left(F^j_t(x)\right) + \lambda\right), \tag{32}$$

where λ is the market price of risk, and Φ is the standard normal cdf. Let the cdf of ${}_n p^j_{x_0}$ under the real-world (physical) probability measure P be

$$F^j_t(s) = Prob_P({}_t p^j_{x_0} \le s). \tag{33}$$

The expected value of ${}_t p^j_{x_0}$ under the Wang risk measure (risk neutral measure), is of the form:

$$E_Q\left[{}_t p^j_{x_0}\right] = \int_0^\infty \left(1 - \tilde{F}^j_t(s)\right) ds. \tag{34}$$

Because ${}_t p^j_{x_0}$ is ranging from 0 to 1, as shown by Denuit et al. (2007), we have

$${}_t E_Q\left[{}_t p^j_{x_0}\right] = \int_0^1 \left(1 - \Phi\left(\Phi^{-1}\left(F^j_t(s)\right) + \lambda\right)\right) ds. \tag{35}$$

The analytical computation of $E_Q\left[{}_t p^j_{x_0}\right]$ according to Eq. (35) is difficult to implement. Therefore, after calibrating the parameters of the time-varying copula model, we use a Monte Carlo simulation with 100,000 iterations to obtain the expected value of the survival probability under the Wang risk measure.

3.3. Valuation of a survivor index swap

To price a survivor index swap, we calculate the fair swap premium, i.e. π. Let M be the face amount of the swap. Under the

[5] For example, the European Investment Bank (EIB) longevity bond is linked to a survival index for English and Welsh men, aged 65 in 2002.

Wang risk measure Q, from the point of view of the hedger, the present value of the payment received by the fixed rate payer for a T-year survival index swap, denoted by V_0^{Fix}, can be calculated as

$$V_0^{Fix} = E_Q\left[\sum_{t=1}^{T}\left(\exp\left(-\int_0^t r(u)\,du\right) \times M \times (1+\pi)S_{x_0}^{ref}(0,t)\right)\right], \tag{36}$$

where $r(t)$ is the risk-free rate.

Furthermore, the present value of the payment received by the floating rate payer for a T-year survival index swap, denoted by V_0^{Float}, is

$$V_0^{Float} = E_Q\left[\sum_{t=1}^{T}\left(\exp\left(-\int_0^t r(u)\,du\right) M \times S_{x_0}(0,t)\right)\right]. \tag{37}$$

Since the present values of the future cash flows received by the fixed rate payer and by the floating rate payer for a survivor index swap shall be equal, we can obtain a fair swap premium by equating Eqs. (36) and (37). Under the assumption that mortality risk and financial risk are independent, we can thus obtain

$$\pi = \frac{\sum_{t=1}^{T}\left(B(0,\ t)\sum_{j=1}^{2N} w_j(0)E_Q\left[{}_t p_{x_0}^j\right]\right)}{\sum_{t=1}^{T}\left(B(0,\ t)\sum_{j=1}^{2N} w_j(0){}_t p_{x_0}^{ref,j}\right)} - 1 \tag{38}$$

where $B(t, T)$ denotes the price of a zero-coupon bond issued at time t that pays \$1 at time T, $t \leq T$.

3.4. Expected shortfall with a survivor index swap

From the standpoint of the pay-fixed payer, we can measure the unexpected loss from the survivor index swap. The loss at time t is defined as the form

$$Loss(t) = M\left((1+\pi)S_{x_0}^{ref}(0,t) - S_{x_0}(0,t)\right), \quad t = 1, \ldots, T. \tag{39}$$

The present value of the total unexpected loss, denoted as PVL, is given by

$$PVL = \sum_{t=1}^{T} B(0,\ t) \times Loss(t) \tag{40}$$

which is determined when the initial value of the swap equals zero, i.e., π, is built in this research. With Eq. (40), we provide the value at risk (VaR) and conditional tail expectation (CTE) of the survivor index swaps.

4. Numerical results

In this section, we first price a survivor index swap. Using the mortality data of Finland, France, the Netherlands and Sweden from 1900 to 2009, we then re-fit the classical LC model with the dynamic copula and non-Gaussian residuals to attain the fair swap premium of the survivor index swap. Finally, we provide the VaR and CTE of the survivor index swaps.

4.1. Model fit of Gaussian and non-Gaussian residuals

To simulate the mortality rates, we first find the optimal lag order of the ARIMA(P, 1, Q) model for each country, according to the Bayesian information criterion (BIC; Schwarz, 1978). The results in Table 2 show that the ARIMA(1, 1, 0) model is the best one for France, whereas the ARIMA(0, 1, 1) model is the best model for Finland, the Netherlands and Sweden. Therefore, we use the best goodness-of-fit ARIMA(P, 1, Q) models for the mortality index for Finland, France, the Netherlands and Sweden.

We then re-fit the error terms of the best goodness-of-fit ARIMA(P, 1, Q) model with several distributions – normal, Student's t, JD, VG, NIG, HYP, GHST, and GH – to model the non-Gaussian property of the error terms from 1900 to 2009. Using mortality data from Finland, France, the Netherlands, and Sweden, we determine the log-likelihood function (LLF), Akaike information criterion (AIC; Akaike, 1974), and BIC results, together with their corresponding ranks, as we summarize in Table 3. All three criteria indicate that the normal distribution provides the worst goodness of fit for all our mortality data, and the GH or JD distributions provide better goodness of fit according

Table 2
BIC values of ARIMA (P, 1, Q) models for each country.

AR order	MA order	Finland	France	Netherlands	Sweden
0	0	187.19	195.36	226.81	162.99
0	1	**183.84**	188.27	**215.64**	**155.16**
0	2	186.19	189.03	217.92	157.23
1	0	184.37	**185.92**	217.67	156.43
1	1	186.19	187.93	217.91	157.32
1	2	187.35	190.20	219.33	158.09
2	0	186.25	188.02	219.12	157.05
2	1	188.53	190.23	220.19	159.40
2	2	189.44	192.17	221.63	161.26

Table 3
Goodness-of-fit tests for the residuals of mortality indices.

Model	LLF	AIC	BIC	**LLF rank**	**AIC rank**	**BIC rank**
Panel A: Mortality index for Finland						
Normal	−176.81	179.81	183.84	8	8	2
T	−173.45	177.45	182.83	7	4	**1**
JD	−171.67	177.67	185.75	**1**	6	7
VG	−172.36	177.36	184.09	5	3	5
NIG	−172.23	177.23	183.96	3	**1**	3
HYP	−172.34	177.34	184.07	4	2	4
GHST	−172.54	177.54	184.26	6	5	6
GH	−172.23	178.23	186.31	2	7	8
Panel B: Mortality index for France						
Normal	−178.89	181.89	185.92	8	8	6
T	−173.28	177.28	182.67	6	4	**1**
JD	−171.96	177.96	186.04	4	6	7
VG	−171.48	176.48	183.21	2	**1**	2
NIG	−171.89	176.89	183.62	3	2	3
HYP	−175.62	180.62	187.35	7	7	8
GHST	−172.20	177.20	183.93	5	3	4
GH	−171.48	177.48	185.55	**1**	5	5
Panel C: Mortality index for the Netherlands						
Normal	−208.60	211.60	215.64	8	8	7
T	−190.93	194.93	200.32	5	3	**1**
JD	−188.52	194.52	202.60	**1**	**1**	4
VG	−192.20	197.20	203.93	6	6	6
NIG	−190.08	195.08	201.81	4	4	3
HYP	−204.26	209.26	215.99	7	7	8
GHST	−189.53	194.53	201.25	3	2	2
GH	**−189.53**	195.53	203.60	2	5	5
Panel D: Mortality index for Sweden						
Normal	−148.13	151.13	155.16	8	8	2
T	−144.96	148.96	154.34	7	3	1
JD	−143.79	149.79	157.86	3	7	8
VG	−143.73	148.73	155.46	2	**1**	3
NIG	−144.13	149.13	155.85	5	4	5
HYP	−143.86	148.86	155.59	4	2	4
GHST	−144.59	149.59	156.32	6	5	6
GH	**−143.73**	149.73	157.80	**1**	6	7

Table 4
Dependence structure for standard residuals of mortality indices.

	Finland	France	Netherlands	Sweden
Panel A: Pearson's linear correlation				
Finland	1	0.5240	0.4239	0.5181
France	0.5240	1	0.5393	0.3090
Netherlands	0.4239	0.5393	1	0.3073
Sweden	0.5181	0.3090	0.3073	1
Panel B: Spearman's rho				
Finland	1	0.4775	0.5008	0.5961
France	0.4775	1	0.5497	0.4154
Netherlands	0.5008	0.5497	1	0.4958
Sweden	0.5961	0.4154	0.4958	1
Panel C: Kendall's tau				
Finland	1	0.3330	0.3551	0.4220
France	0.3330	1	0.3982	0.2926
Netherlands	0.3551	0.3982	1	0.3663
Sweden	0.4220	0.2926	0.3663	1

to LLF. However, because the BIC introduces a penalty term for the effective number of parameters, this criterion indicates that Student's t model offers the best fit for mortality data from Finland, France, the Netherlands, and Sweden. Thus, the goodness-of-fit tests consistently indicate that non-Gaussian distributions provide better in-sample for the error terms of the mortality indices.

Using the best in-sample goodness of fit for the standard residuals of the mortality indices, we report, in Table 4, their dependence measures according to linear correlation, Spearman's rho and Kendall's tau. During 1900–2009 period, we find that the mortality rates exhibit mortality dependence. For example, Pearson's linear correlations are all positive and range from 0.3073 to 0.5393, which means that the positive mortality dependence structure creates a serious problem for pricing survivor index swaps linked to a cohort survival index with cross-country mortality improvement.

4.2. Selection of the dynamic copula model

The probability integral transforms for the standard residuals obtained from the best goodness-of-fit models provide the input data for estimating the static and dynamic copulas. As Table 5 shows, compared with the static copula models, the dynamic copula models provide better goodness of fit, in terms of LLF. However, for the Gaussian, Student's t, and skewed t copulas, the DCC version does not provide a significant improvement in LLF, such that its goodness of fit is even worse that the fit of the static version, according to both AIC and BIC. Therefore, for Gaussian, Student's t, and skewed t copulas, it is sufficient to use static Student's t copulas to model mortality dependence, according to the BIC. For Clayton and Gumbel copulas, the time-varying version provides better goodness of fit, as assessed by LLF, AIC, and BIC. In addition, among all competing copulas, the time-varying Gumbel copula provides the best goodness-of-fit performance, according

Table 6
Parameters of the time-varying Gumbel copula.

Parameters	Estimate	Standard error	t-statistic
ω_θ	−0.4027	0.1158	−3.48
$\gamma_{1,2}^{\theta}$	−1.0103	0.2480	−4.07
$\gamma_{1,3}^{\theta}$	2.7186	0.4866	5.59
$\gamma_{1,4}^{\theta}$	−0.6321	0.2273	−2.78
$\gamma_{2,3}^{\theta}$	0.7077	0.0867	8.17
$\gamma_{2,4}^{\theta}$	1.0376	0.3482	2.98
$\gamma_{3,4}^{\theta}$	−0.7627	0.0752	−10.14
β_θ	1.0572	0.0159	66.56
θ_0	1.5258	0.1318	11.58

Notes: This table provides the parameter estimates and robust standard errors of the time-varying Gumbel copula. The t-statistic is the ratio of the parameter estimate to its corresponding robust standard error. θ_0, ω_θ, β_θ and $\gamma_{i,j}^{\theta}$ are defined in Eq. (R.10), where i or/and j equal 1 for Finland, 2 for France, 3 for the Netherlands and 4 for Sweden.

to LLF, AIC, and BIC. The parameter estimates of the time-varying Gumbel copula are in Table 6. On the basis of these results, we chose to examine the fair swap rates, VaR, and CTE using the time-varying Gumbel copula model, with Student's t residuals, to simulate mortality rates for Finland, France, the Netherlands, and Sweden.

We provide a numerical example of the survivor index swaps for a cohort of 65-year-old persons in the calendar year 2009. The initial term structure is obtained from the US Department of the Treasury. We assume that the exchange rates are equal to 1 and the initial annuities issued to each of the four mortality group equal 0.25. Table 7 reveals the fair swap premiums, with time to maturity equal to 25 years when is −0.1, −0.15, and −0.2, and with parallel shifts upward of 0%, 2%, and 4% in the yield curve. From Table 7, we demonstrate that the lower the λ and the interest rate are, the higher is the fair swap premium. Similarly, the fair swap premiums of the DCC copula models with non-Gaussian residuals are higher than those of the original LC model, even when the yield curve moves up in parallel. From Table 8, we reach the same conclusion by varying the time to maturity.

Table 9 presents the VaR and CTE of the *PVL* with maturation times of up to 25 years. Compared with the original LC model, the best prediction model has higher VaR and CTE values. Because shorter duration contracts cover less longevity risk, the VaR and CTE values are smaller for shorter duration survivor index swaps. The differences between the original LC and the best prediction models instead are greater for longer durations.

5. Conclusion

As life expectancy continues to increase, longevity risk has become a global phenomenon. The use of capital market solutions such as survivor swaps and longevity bonds to hedge longevity risk consequently has become more important as well. To increase hedging effectiveness, it is necessary to deal with basis risk for the development of mortality-linked securities. This research proposes

Table 5
Estimation of copula models for mortality indices.

Model	LLF	AIC	BIC	LLF rank	AIC rank	BIC rank
Clayton copula	46.35	−45.35	−44.01	10	10	10
Time-varying Clayton copula	74.40	−65.40	−53.29	4	2	3
Gumbel copula	51.12	−50.12	−48.77	9	9	8
Time-varying Gumbel copula	85.67	−76.67	−64.56	1	1	1
Gaussian copula	65.44	−59.44	−51.37	8	7	4
DCC Gaussian copula	65.44	−58.44	−49.02	7	8	7
Student's t copula	72.24	−65.24	−55.82	6	3	2
DCC Student's t copula	72.35	−63.35	−51.24	5	6	5
Skewed t copula	76.22	−65.22	−50.42	3	4	6
DCC Skewed t copula	76.76	−64.76	−48.62	2	5	9

Table 7
Swap premiums for different interest rates (units: bps).

Yield Rates	Model	$\lambda = -0.1$	$\lambda = -0.15$	$\lambda = -0.2$
Original yield curve	Time-varying Gumbel + non-Gaussian	131.31	145.23	159.10
	No dependence + Normal	126.08	140.71	155.29
Parallel shift up of 2%	Time-varying Gumbel + non-Gaussian	102.10	113.99	125.84
	No dependence + Normal	97.63	110.14	122.60
Parallel shift up of 4%	Time-varying Gumbel + non-Gaussian	78.48	88.64	98.77
	No dependence + Normal	74.67	85.36	96.01

Notes: Time to maturity is 25 years. "Time-varying Gumbel + non-Gaussian" indicates that we use the time-varying Gumbel copula model to capture the dependence structure of the multi-country mortality indices, applying the best goodness-of-fit Student's t distribution to model the residuals of the mortality index. "No dependence + Normal" means that we ignore the dependence structure of the multi-country mortality indices and use the Gaussian distribution to model the residuals of the mortality index.

Table 8
Swap premiums for different maturities (units: bps).

Time to maturity	Model	$\lambda = -0.1$	$\lambda = -0.15$	$\lambda = -0.2$
15	Time-varying Gumbel + non-Gaussian	12.99	19.87	26.70
	No dependence + Normal	10.44	17.68	24.88
20	Time-varying Gumbel + non-Gaussian	62.49	73.05	83.57
	No dependence + Normal	58.45	69.56	80.62
25	Time-varying Gumbel + non-Gaussian	131.31	145.23	159.10
	No dependence + Normal	126.08	140.71	155.29

Notes: "Time-varying Gumbel + non-Gaussian" indicates that we use the time-varying Gumbel copula model to capture the dependence structure of the multi-country mortality indices, applying the best goodness-of-fit Student's t distribution to model the residuals of the mortality index. "No dependence + Normal" means that we ignore the dependence structure of the multi-country mortality indices and use the Gaussian distribution to model the residuals of the mortality index.

Table 9
VaR and CTE of the losses for different maturation times ($\lambda = -0.1$).

Time to maturity	Model	VaR95	VaR99	CTE95	CTE99
15	Time-varying Gumbel + non-Gaussian	0.1600	0.2235	0.1991	0.2569
	No dependence + Normal	0.1080	0.1519	0.1349	0.1720
20	Time-varying Gumbel + non-Gaussian	0.2742	0.3857	0.3419	0.4404
	No dependence + Normal	0.1850	0.2615	0.2318	0.2963
25	Time-varying Gumbel + non-Gaussian	0.3799	0.5354	0.4754	0.6150
	No dependence + Normal	0.2542	0.3622	0.3209	0.4135

Notes: "Time-varying Gumbel + non-Gaussian" indicates that we use the time-varying Gumbel copula model to capture the dependence structure of the multi-country mortality indices, applying the best goodness-of-fit Student's t distribution to model the residuals of the mortality index. "No dependence + Normal" means that we ignore the dependence structure of the multi-country mortality indices and use the Gaussian distribution to model the residuals of the mortality index.

survivor index swaps with a cohort survivor index that can serve as an effective hedging instrument for both the annuity provider and the pension provider. To price the survivor index swaps, this study investigates, for the first time, time-varying mortality dependence across countries, as well as non-Gaussian residuals under the multi-country Lee–Carter framework.

By providing illustrations based in the male mortality experience in four countries, Finland, France, the Netherlands, and Sweden, we uncover consistent support for the non-Gaussian residuals and time-varying mortality dependence across countries. Specifically, when we calibrate the parameters of the RH model, Student's t model is the best one for all mortality indices, according to the BIC criterion. For the mortality projection for these five countries, we find that the normal distribution provides weak mortality in-sample performance, whereas the non-Gaussian distributions provide good in-sample performance. In the survivor index swap application, we demonstrate that the swap curves of the original LC model are lower than those of the LC model with time-varying dependence and non-Gaussian innovations. In addition, the VaR and CTE values of the original LC model are lower than those of the best goodness-of-fit model. Choosing an appropriate leptokurtic model is critical to avoiding an underestimation of the loss for the insurer due to longevity risk. The contributions of this research thus are significant not only for academic purposes but also for the development of real-world capital market solutions to hedge longevity risk.

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Appendix. The IFM method for mortality index

To model the mortality dependence across N countries, under the time-varying copula model, the conditional N-dimensional cumulative distribution function (cdf) of ε_t^j is as follows:

$$P\left(\varepsilon_t^1 \le x_{1t}, \ldots, \varepsilon_t^N \le x_{Nt} \middle| \Im_{t-1}\right) = C\left(F_1\left(x_{1t} \middle| \Im_{t-1}\right), \ldots, F_N\left(x_{Nt} \middle| \Im_{t-1}\right) \middle| \Im_{t-1}\right), \tag{A.1}$$

where ε_t^j is the error term of the ith mortality index and $x_t = \left\{x_{1t}, \ldots, x_{Nt}\right\}$ is the realized residual vector in the tth year. To estimate the parameters of the mortality index, we adopt the inference for margins (IFM) method proposed by Joe and Xu (1996) and Joe (1997). Let $\Theta = \{\Theta_M, \Theta_C\}$, where $\Theta_M = \left\{\Theta_1^M, \ldots, \Theta_N^M\right\}$ is the marginal parameter set; Θ_j^M is the parameter set of the jth margin; and Θ_C is copula parameter set. The IFM method is a two-step estimation procedure. The first step is equivalent to N estimations of univariate mortality indices. We estimate each margin's parameter set Θ_j^M, $j = 1, \ldots, N$, by performing the estimation of the marginal distributions.

$$\hat{\Theta}_j^M = \arg\max_{\Theta_j^M} \sum_{t=1}^{T} \ln f_i \left(x_{it} | \Im_{t-1}; \Theta_j^M \right). \tag{A.2}$$

Consequently, for each country, we calibrate the parameters of mortality index in Eq. (3) and its error terms in Eqs. (4) and (5) in the first step.

In the second step, given the calibrated marginal parameter set $\hat{\Theta}_M = \left\{ \hat{\Theta}_1^M, \ldots, \hat{\Theta}_N^M \right\}$ obtained in the first step, we can estimate the (time-varying) copula parameter set in the following:

$$\hat{\Theta}_C = \arg\max_{\Theta_C} \sum_{t=1}^{T} \log \left(c \left(F_1(x_{1t}), \ldots, F_N(x_{Nt}) \Im_{t-1}; \hat{\Theta}_M; \Theta_C \right) \right). \tag{A.3}$$

The IFM estimator is defined as the vector

$$\hat{\Theta} = \left(\hat{\Theta}_M, \hat{\Theta}_C \right)'. \tag{A.4}$$

This completes the two-step estimation procedure for the IFM method.

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